Theorems
Algebraic topology qualifying course
MSU, Spring 2017

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This document was made as a way to study the material from the spring semester algebraic topology qualifying course at Michigan State University, in spring of 2017. It serves as a companion document to the "Definitions" review sheet for the same class. The main textbook for the course was *Algebraic Topology*, by Hatcher.

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1 Homotopy and Cell Complexes

Proposition 1.1. *Homotopy of maps is an equivalence relation.*

Proposition 1.2. *Homotopy of spaces is an equivalence relation.*

Proposition 1.3. *Two spaces X, Y are homotopy equivalent if and only if there is a space Z containing both X and Y such that X, Y are both deformation retracts of Z.*

Proposition 1.4. *A contractible space is path connected.*

Lemma 1.5. *Let $f_0, f_1 : X \to Y$ be homotopic and $g : Y \to Z$. Then $gf_0 \simeq gf_1$.*

Lemma 1.6. *Let $f_0, f_1 : X \to Y$ be homotopic and $h : Z \to X$. Then $f_0h \simeq f_1h$.*

Theorem 1.7. *If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \to X/A$ is a homotopy equivalence. That is, if a subcomplex is contractible, we can contract it to a point without changing the homotopy class of X.*

Theorem 1.8. *Let X, Y be CW complexes. Then $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$.*

Theorem 1.9. *Let (X, A) be a CW pair and $f, g : A \to Y$ be homotopic. Then $Y \sqcup_f X \simeq Y \sqcup_g X$. That is, homotopic attaching maps form homotopy equivalent spaces after attaching.*

Theorem 1.10. *A pair (X, A) has the homotopy extension property if and only if $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$.*

Theorem 1.11. *Let (X, A) be a CW pair. Then $(X \times \{0\}) \cup (A \times I)$ is a deformation retract of $X \times I$. Consequently, (X, A) has the homotopy extension property.*

Theorem 1.12. *If (X, A) satisfies the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Theorem 1.13. *Let (X, A) be a CW pair, and $f, g : A \to Y$ be attaching maps with $f \simeq g$. Then $Y \sqcup_f X \simeq Y \sqcup_g X \operatorname{rel} Y$.*

Theorem 1.14. *Suppose (X, A) and (Y, A) satisfy the homotopy extension property and $f : X \to Y$ is a homotopy equivalence with $f|_A = \mathrm{Id}_A$. Then f is a homotopy equivalence $\operatorname{rel} A$.*

Theorem 1.15. *If (X, A) satisfies the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.*

Theorem 1.16. *A map $f : X \to Y$ is a homotopy equivalence if and only if X is a deformation retract of the mapping cylinder M_f. That is, X, Y are homotopy equivalent if and only if there is a space containing both X, Y as deformation retracts.*

2 The Fundamental Group

Proposition 2.1. *Let X be a space and fix $x_0, x_1 \in X$. The relation of homotopy of paths $f : I \to X$ with $f(0) = x_0$ and $f(1) = x_1$ is an equivalence relation.*

Proposition 2.2. *Let f_0, f_1, g_0, g_1 be paths such that $f_0(1) = g_0(0)$ and $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.*

Proposition 2.3. *$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.*

Proposition 2.4. *A change-of-basepoint map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

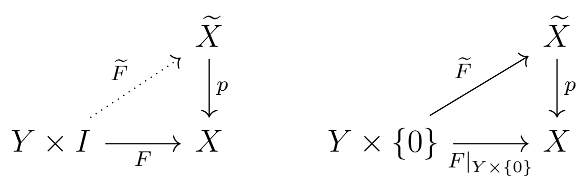
Proposition 2.5. *A space is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X.*

Proposition 2.6. *$\pi_1(S^1) \cong \mathbb{Z}$. In particular, it is generated by the homotopy class of the loop $w : I \to S^1$ given by $t \mapsto e^{2\pi i t}$.*

Proposition 2.7. *Let $p : \widetilde{X} \to X$ be a covering map. For each path $f : I \to X$ starting at $x_0 \in X$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ starting at $\widetilde{x}_0$.*

Proposition 2.8. *Let $p : \widetilde{X} \to X$ be a covering map. For each homotopy of paths $f_t : I \to X$ starting at x_0 and each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy of pahts $\widetilde{f}_t : I \to \widetilde{X}$ starting at $\widetilde{x}_0$.*

Proposition 2.9. *Let $p : \widetilde{X} \to X$ be a covering map. Let $F : Y \times I \to X$ and suppose there is a map $\widetilde{F} : Y \times \{0\} \to \widetilde{X}$ that is a lift of $F|_{Y \times \{0\}}$. Then there is a unique map $\widetilde{F} : Y \times I \to X$ lifting F that restricts to the previous $\widetilde{F}$ on $Y \times \{0\}$. Diagrammatically, given all of the following solid arrows, the dotted arrow exists and is the unique map making the triangle commute.*



2.1 Corollaries to $\pi_1(S^1) \cong \mathbb{Z}$

Proposition 2.10 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

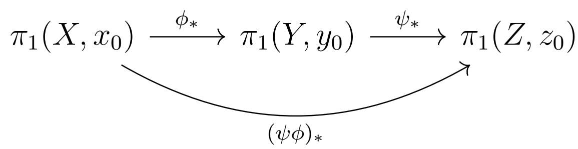
Proposition 2.11 (Brouwer Fixed Point Theorem, dimension 2)**.** *Every continuous map $h : D^2 \to D^2$ has a fixed point.*

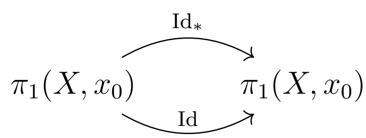
Proposition 2.12 (Borsuk-Ulam Theorem, dimension 2)**.** *Let $f : S^2 \to \mathbb{R}^2$ be continuous. Then there exists $x \in S^2$ so that $f(x) = f(-x)$.*

Proposition 2.13. *If $S^2 = A_1 \cup A_2 \cup A_3$ where A_i are closed, then one A_i must contain a pair of antipodal points.*

Proposition 2.14. *Let X, Y be path connected. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Proposition 2.15. *The assignment $(X, x_0) \mapsto \pi_1(X, x_0)$ and $\phi \mapsto \phi_*$ is a functor from the category of pointed topological spaces to the category of groups. That is, if we have pointed spaces $(X, x_0), (Y, y_0), (Z, z_0)$ and pointed continuous maps $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (Z, z_0)$, the following two diagrams commute.*





That is, $(\psi\phi)* = \psi_*\phi_*$ and $\mathrm{Id}_* = \mathrm{Id}_{\pi_1(X,x_0)}$.

Proposition 2.16. $\pi_1(S^n) = 0$ *for $n \geq 2$.*

Proposition 2.17. *Let $X = \bigcup_\alpha A_\alpha$, where each A_α is a path connected open subset of X, and $x_0 \in A_\alpha$ for all α. Suppose that $A_\alpha \cap A_\beta$ is path connected for each α, β. Then every loop in X based at x_0 is homotopic to a product of loops each of which is contained in some A_α.*

Proposition 2.18. $\mathbb{R}^2$ *is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*

Proposition 2.19. *Let A be a retract of X and $\iota : A \to X$ be the inclusion. Then the induced homomorphism $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective. If A is a deformation retract of X, then ι_* is an isomorphism.*

Proposition 2.20. *Let $r : X \to A$ be a retraction. Then $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.*

Proposition 2.21. *If $\phi : X \to Y$ is a homotopy equivalence, then $\phi_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

2.2 Van Kampen's Theorem

Proposition 2.22 (Universal Property of Free Product)**.** *Let G_α be a collection of groups, and let $*_\alpha G_\alpha$ be the free product. Let H be a group, and suppose we have a collection $\phi_\alpha : G_\alpha \to H$ of group homomorphisms. Then there is a unique homomorphism $\phi : *_\alpha G_\alpha \to H$. Given a word $g_1 \dots g_n$, ϕ acts on the word by applying ϕ_α to each g_i, where α is chosen to match which G_α that particular g_i comes from.*

Proposition 2.23 (Van Kampen's Theorem)**.** *Let $X = \bigcup_\alpha A_\alpha$ where each A_α is path connected and an open subset of X, and each A_α contains the basepoint x_0. Suppose that $A_\alpha \cap A_\beta$ is path connected for all α, β. Let $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ be the homomorphism induced* *by the inclusion $A_\alpha \hookrightarrow X$. Let $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ be the unique extension of all the j_α. Then Φ is surjective.*

Let $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(X, x_0)$ be the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$. If $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected for all α, β, γ, then $\ker \Phi$ is the normal subgroup N generated by elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$. Consequently,

$$\pi_1(X, x_0) \cong (*_\alpha \pi_1(A_\alpha))\,/N$$

Proposition 2.24. *Let (X_α, x_α) be a collection of path connected pointed spaces, so that for each α there exists $U_\alpha \subset X_\alpha$ where U_α deformation retracts to the point x_α. Then (by Van Kampen's Theorem), the fundamental group of the wedge sum by identifying all basepoints x_α is the free product of the fundamental groups. Symbolically,*

$$\pi_1 \left(\bigvee_\alpha X_\alpha \right) \cong *_\alpha X_\alpha$$

Proposition 2.25.

$$\pi_1 \left(\bigvee_{\alpha \in A} S^1_\alpha \right) \cong *_\alpha \mathbb{Z} \cong F\langle A \rangle$$

where $F\langle A \rangle$ is the free group on the set A.

Proposition 2.26. *The fundamental group of any connected graph is free.*

Proposition 2.27. *Let (X, x_0) be a path connected space, and form a space (Y, x_0) by attaching 2-cells e^2_α to X via attaching maps $\phi_\alpha : (S^1, s_0) \to (X, x_0)$. Then the inclusion $\iota : (X, x_0) \hookrightarrow (Y, x_0)$ induces a surjection $\iota_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$.*

Then kernel of ι_* is the subgroup of $\pi_1(X, x_0)$ generated by elements of the form $\gamma_\alpha \phi_\alpha \overline{\gamma}_\alpha$, where ϕ_α is one of the attaching maps and γ_α is a path from x_0 to $\phi_\alpha(s_0)$.

$$\ker \iota_* = \langle \gamma_\alpha \phi_\alpha \overline{\gamma}_\alpha \rangle$$

Consequently, $\pi_1(Y) \cong \pi_1(X)/\ker \iota_*$.

Proposition 2.28. *Let (X, x_0) be a path connected space, and form a space (Y, x_0) by attaching n-cells to X, where $n > 2$. Then the inclusion $(X, x_0) \hookrightarrow (Y, x_0)$ induces an isomorphism $\pi_1(X, x_0) \cong \pi_1(Y, x_0)$.*

Proposition 2.29. *Let X be a path connected cell complex. Then the inclusion of the 2-skeleton $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \cong \pi_1(X)$.*

Proposition 2.30. *Let M_g be the orientable surface of genus g. Then $\pi_1(M_g)$ has the presentation*

$$\langle\ a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1}, \ldots, a_g b_g a_g^{-1} b_g^{-1}\ \rangle$$

As a consequence, M_g is homotopy equivalent to M_h if and only if $g = h$.

Proposition 2.31. *Let N_g be the nonorientable surface of genus g. Then $\pi_1(N_g)$ has the presentation*

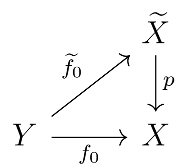
$$\langle\, a_1, \ldots, a_g \mid a_1^2 \ldots a_g^2 \,\rangle$$

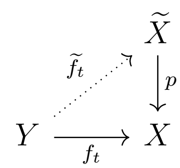
As a consequence, N_g is homotopy equivalent to N_h if and only if $g = h$.

Proposition 2.32. *Let G be a group. Then there is a 2-dimensional cell complex X so that $\pi_1(X) \cong G$.*

2.3 Covering Spaces

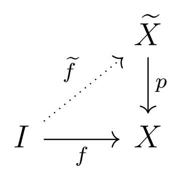
Proposition 2.33 (Homotopy Lifting Property)**.** *Let $p : \widetilde{X} \to X$ be a covering space, and let $f_t : Y \to X$ be a homotopy, and suppose there is a lift $\widetilde{f_0} : Y \to \widetilde{X}$ lifting f_0. Then there exists a unique homotopy $\widetilde{f_t} : Y \to \widetilde{X}$ lifting f_t that agrees with $\widetilde{f_0}$ at $t = 0$. This is depicted in the following diagrams.*





The following is a special case of the homotopy lifting property where Y is a single point.

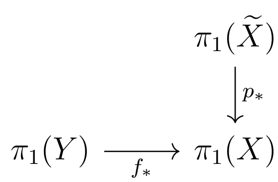
Proposition 2.34 (Path Lifting Property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $f : I \to X$ be a path. Define $x_0 = f(0)$. For each $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{f} : I \to \widetilde{X}$ lifting f that begins at $\widetilde{x}_0$.*

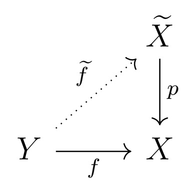


Proposition 2.35. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map. The induced map $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective. Furthermore, the image subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ consists of homotopy classes of loops in X based at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x}_0$ are loops.*

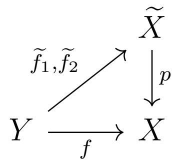
Proposition 2.36. *Let $p : \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ and X path connected. The number of sheets of the covering is equal to the index of the image subgroup $p_*(\pi_1(\widetilde{X}))$ in $\pi_1(X)$.*

Proposition 2.37 (Lifting Criterion)**.** *Let $p : \widetilde{X} \to X$ be a covering space, and $f : Y \to X$ be a map. Assume that Y is path connected and locally path connected. Then a lift $\widetilde{f} : Y \to \widetilde{X}$ exists if and only if $f_*(\pi_1(Y)) \subset p_*(\pi_1(\widetilde{X}))$. (Below are some relevant diagrams.)*





Proposition 2.38 (Unique Lifting Property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and let $f : Y \to X$, where Y is connected. If $\widetilde{f_1}, \widetilde{f_2}$ are both lifts of f, and $\widetilde{f_1}$ and $\widetilde{f_2}$ agree on one point, then $\widetilde{f_1} = \widetilde{f_2}$.*



Proposition 2.39. *Let X be a path connected, locally path connected, and semilocally simply connected. Then there is a simply connected covering space $p : \widetilde{X} \to X$. It is unique up to isomorphism.*

Proposition 2.40. *Let X be path connected, locally path connected, and semilocally simply connected. Then for every subgroup $H \subset \pi_1(X)$, there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H)) = H$.*

Proposition 2.41. *Let (X, x_0) be path connected and locally path connected. Let $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X_2$ be path connected covering spaces. Then $\widetilde{X}_1$ and $\widetilde{X}_2$ are isomorphic via an isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ taking a basepoint $\widetilde{x}_1 \in p_1^{-1}(x_0)$ to $\widetilde{x}_2 \in p_2^{-1}(x_0)$ if and only if $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

Proposition 2.42 (Classification of Covering Spaces, with basepoints)**.** *Let (X, x_0) be path connected, locally path connected, and semilocally simply connected. To a path connected covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, we associate the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. This gives a bijection between the set of basepoint-preserving isomorphism classes of path connected covering spaces $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$.*

Proposition 2.43 (Classification of Covering Spaces, ignoring basepoints)**.** *Let X be path connected, locally path connected, and semilocally simply connected. To a path connected covering space $p : \widetilde{X} \to X$, we associate the conjugacy class of the subgroup $p_*(\pi_1(\widetilde{X}))$ in $\pi_1(X)$. This gives a bijection between isomorphism classes of path connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X)$.*

Proposition 2.44. *Let $p : \widetilde{X} \to X$. A deck transformation $\widetilde{X} \to \widetilde{X}$ is uniquely determined by where it sends a single point. In particular, any deck transformation with a fixed point is the identity.*

Proposition 2.45. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a path connected covering space of a path connected and locally path connected space X. Let H be the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Then*

1. *The covering space $p : \widetilde{X} \to X$ is normal if and only if H is a normal subgroup.*

2. *The group of deck transformations is isomorphic to the quotient $N(H)/H$, where $N(H)$ is the normalizer.*

3. *If $\widetilde{X}$ is the universal cover, then the group of deck transformations is isomorphic to $\pi_1(X)$. (This follows immediately from (2)).*

Proposition 2.46. *Let G act on $\widetilde{X}$ via a covering space action. Then*

1. *The quotient map $p : \widetilde{X} \to \widetilde{X}/G$ defined by $\widetilde{x} \mapsto G\widetilde{x}$ is a normal covering space.*
2. *If $\widetilde{X}$ is path connected, then G is the group of deck transformations of the covering space $p : \widetilde{X} \to \widetilde{X}/G$.*
3. *If $\widetilde{X}$ is path connected and locally path connected, then $G \cong \pi_1(\widetilde{X}/G)/p_*(\pi_1(\widetilde{X}))$.*
4. *If $\widetilde{X}$ is path connected, locally path conected, and simply connected, then $G \cong \pi_1(\widetilde{X}/G)$.*

3 Simplicial and Singular Homology

3.1 Δ-Complexes and Simplicial Homology

Proposition 3.1. *Let X be a Δ-complex, and let ∂_n be the boundary homomorphisms. Then $\partial_{n-1} \circ \partial_n = 0$. Thus we have a chain complex*

$$\dots \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X) \xrightarrow{\partial_{n-2}} \dots$$

3.2 Singular Homology

Proposition 3.2. *Let X be a space, and let ∂_n be the boundary homomorphism. Then $\partial_{n-1} \circ \partial_n = 0$, so we have a chain complex*

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \dots$$

Proposition 3.3. *Homeomorphic spaces have isomorphic singular homology groups.*

Proposition 3.4. *Let X be a space, written as a disjoint union $\bigsqcup_\alpha X_\alpha$ where X_α are the path components of X. Then*

$$H_n(X) = H_n\left(\bigsqcup_\alpha X_\alpha\right) \cong \bigoplus_\alpha H_n(X_\alpha)$$

Proposition 3.5. *If X is nonempty and path connected, then $H_0(X) \cong \mathbb{Z}$. Consequently, for any space X, $H_0(X) = \bigoplus_{i \in I} \mathbb{Z}$ where the path components of X are also indexed by I.*

Proposition 3.6. *If X is a point, then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n \geq 1$.*

Proposition 3.7. *Let X be a path connected space. Then $H_1(X)$ is the abelianization of $\pi_1(X)$.*

3.3 Homotopy Invariance of Singular Homology

Proposition 3.8. *Homotopy equivalent spaces have isomorphic homology groups. (To be shown at the end of this section.)*

Proposition 3.9. *Let $f : X \to Y$, and let $f_\# : C_n(X) \to C_n(Y)$ be the induced map on singular n-chains. Then $f_\# \partial = \partial f_\#$, so the following diagram commutes.*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \dots \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\ \dots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \longrightarrow & \dots \end{array}$$

Proposition 3.10. *A chain map of chain complexes induces homomorphisms between the homology groups of the complexes. Concretely, if C_n and D_n are complexes with homology H_n^C and H_n^D respectively, and $f_n : C_n \to D_n$ is a chain map, then f_n induces a homomorphism $f_n : H_n^C \to H_n^D$.*

Proposition 3.11 (Functoriality of Homology)**.** *Let $f : Y \to Z$ and $g : X \to Y$, and let f_*, g_* be the induced maps on (singular) homology. Then $(fg)_* = f_* g_*$. Additionally, the induced map from the identity is the identity, i.e. $(\mathrm{Id}_X)_* = \mathrm{Id}_{H_n(X)}$.*

Proposition 3.12. *Chain homotopic maps induce the same homomorphism on homology.*

Proposition 3.13. *If $f, g : X \to Y$ are homotopic, then the induced maps $f_\#, g_\# : C_n(X) \to C_n(Y)$ are chain homotopic. Consequently, f, g induce the same homomorphism on homology, $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proposition 3.14. *If $f : X \to Y$ is a homotopy equivalence, then the induced maps $f_* : H_n(X) \to H_n(Y)$ are isomorphisms for all n.*

Proposition 3.15 (Some Basic Facts About Exact Sequences)**.** *Let A, B, C be abelian groups. (Or more generally, R-modules.)*

1. *$0 \to A \to B$ is exact if and only if $A \to B$ is injective.*
2. *$B \to C \to 0$ is exact if and only if $B \to C$ is surjective.*
3. *$0 \to A \to B \to 0$ is exact if and only if $A \to B$ is an isomorphism.*
4. *$0 \to A \to B \to C \to 0$ is exact if and only if $A \to B$ is injective and $B \to C$ is surjective. If this is exact, then $B \to C$ induces an isomorphism $C \cong B/\operatorname{im} A$*

Proposition 3.16 (Long Exact Sequence of Homology)**.** *Let X be a space and let $A \subset X$ be a nonempty closed subspace that is a deformation retract of some neighborhood in X. (That is, (X, A) is a good pair.) Then there is an exact sequence*

$$\dots \longrightarrow \widetilde{H}_n(A) \xrightarrow{\iota_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{\iota_*} \widetilde{H}_{n-1}(X) \longrightarrow \dots$$

$$\dots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$$

Proposition 3.17 (Homology of Spheres). *$\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_i(S^n) = 0$ for $i \neq n$.*

Proposition 3.18 (Brouwer Fixed Point Theorem). *∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed point.*

Proposition 3.19. *Let X be a space and $A \subset X$. Then we have an exact sequence of chain complexes $0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$. This induces a long exact sequence on homology:*

$$\dots \longrightarrow H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{\iota_*} H_{n-1}(X) \longrightarrow \dots$$

$$\dots \longrightarrow H_0(X, A) \longrightarrow 0$$

Proposition 3.20. *If $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.*

Proposition 3.21 (Excision Theorem, Version 1). *Let X be a space and let $Z \subset A \subset X$ where the closure of Z is contained in the interior of A. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ for all n.*

Proposition 3.22 (Excision Theorem, Version 2). *Let X be a space and let $A, B \subset X$ such that the $X \subset \mathring{A} \cup \mathring{B}$. Then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.*

Proposition 3.23 (Mayer-Vietoris Sequence). *Let X be a space. If $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$, then we have an exact sequence on homology:*

$$\dots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \dots$$

$$\dots \longrightarrow H_0(X) \longrightarrow 0$$

More generally, if $X = A \cup B$ and there are neighborhoods U such that U deformation retracts onto A and V where V deformation retracts onto B, then we get the same exact sequence. (So we can not worry about whether $X = \mathring{A} \cup \mathring{B}$.)